TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 196, 1974

SU(n) ACTIONS ON DIFFERENTIABLE MANIFOLDS WITH VANISHING FIRST AND SECOND INTEGRAL PONTRJAGIN CLASSES[1]

BY

EDWARD A. GROVE

ABSTRACT. In this paper we determine the connected component of the identity of the isotropy subgroups of a given action of SU(n) on a connected manifold whose first and second integral Pontrjagin classes are zero and whose dimension is less than $n^2 - 8n/3 - 1$.

1. **General introduction.** Let SU(n) be a differentiable transformation group acting on a connected manifold M with $\dim M \leq n^2 - 8n/3 - 2$, and assume that the first two integral Pontrjagin classes of M are zero. The purpose of this paper is to determine the orbit structure of M under the action of SU(n).

To be more precise, in Theorem 2.3 we determine those closed connected simple subgroups H of SU(n) such that the first integral Pontrjagin class of SU(n)/H, $P_1(\mathrm{SU}(n)/H; Z)$, is zero; in Theorem 2.5 we determine those closed connected simple subgroups H of SU(n) such that $P_1(\mathrm{SU}(n)/H; Z) = 0 = P_2(\mathrm{SU}(n)/H; Z)$; in Theorem 2.9 we determine the connected component of the identity of a regular isotropy subgroup of the given above actions of SU(n) on M. In Theorem 3.1 we then show how the connected component of the identity of all isotropy subgroups of SU(n) is completely determined by the connected component of the identity of the principal isotropy subgroup, and we give a list of such possible subgroups.

Wu-Chung Hsiang and Wu-Yi Hsiang considered this problem in [4]. One of the key steps in their approach was their knowledge of large subgroups of SU(n). This forced them to put a severe restriction on M. Our approach is more general in nature, and allows us a much less restrictive dimension requirement.

The author wishes to thank his thesis advisor Wu-Yi Hsiang for the encouragement and help he received while writing this paper. Some of these results were included in the author's Ph.D. thesis at Brown University.

In this section we recall several well-known facts. Let G be a compact con-

Received by the editors June 23, 1970 and, in revised form, July 6, 1973.

AMS (MOS) subject classifications (1970). Primary 57E15; Secondary 57E25.

Key words and phrases. Compact transformation groups, integral Pontrjagin classes, isotropy subgroups.

(1) Research supported in part by National Science Foundation Grant GP-6371.

nected Lie group acting differentiably on a manifold M. For each $x \in M$, let $G_x = \{g \in G : g \cdot x = x\}$ and let $G(x) = \{g \cdot x : g \in G\}$. Then if $f : G/G_x \approx G(x) \subsetneq M$ is the natural map,

$$f\,\tau(M) = \tau(G/G_x) \oplus \nu(G/G_x) = \alpha_{\xi_1}(\iota_{G/G_x}) \oplus \alpha_{\xi_1}(\phi_x)$$

where ξ_1 is the principal bundle $(G_x \to G \to G/G_x)$, ι_{G/G_x} is the isotropy representation of G_x in G, and ϕ_x is the slice representation of G_x at x. Suppose now that $x \in M$ is a regular element. Then ϕ_x is a trivial representation [5], and so $f^!\tau(M) = \alpha_{\xi_1}(\iota_{G/G_x}) \oplus \theta$ where θ is a trivial vector bundle. The following result is trivial to prove but is frequently useful.

Proposition 1.1. *Let $\eta = (H \to E \to B)$ and $\eta' = (G \to E' \to B')$ be principal bundles, and let*

$$\begin{array}{ccc} G & \overset{\Lambda}{\dashrightarrow} & G' \\ \downarrow & & \downarrow \\ E & \dashrightarrow & E' \\ \downarrow & & \downarrow \\ B & \overset{x}{\dashrightarrow} & B' \\ \vdots & & \vdots \\ \eta & & \eta' \end{array}$$

be a homomorphism of principal bundles. Then

$$\begin{array}{ccc} R(G) & \overset{\Lambda^!}{\longleftarrow} & R(G') \\ \downarrow{\scriptstyle \alpha_\eta} & & \downarrow{\scriptstyle \alpha_{\eta'}} \\ K(B) & \overset{x^!}{\longleftarrow} & K(B') \end{array}$$

is a commutative diagram.

So if $F : G/G_x^0 \to G/G_x$ is the natural map and ξ_2 is the principal bundle $(G_x^0 \to G \to G/G_x^0)$, we have

$$F^! f^! \tau(M) = \alpha_{\xi_2}(\iota_{G/G_x^0}) \oplus \theta = \tau(G/G_x^0) \oplus \theta,$$

and so $F^*(f^*(P_j(M; Z))) = P_j(G/G_x^0; Z)$ (where $P_j(N; Z)$ is the jth integral Pontrjagin class of N).

So to study G actions on a manifold M satisfying $P_1(M; Z) = 0 = P_2(M; Z)$, we must first find all those closed connected subgroups H of G such that $P_1(G/H; Z) = 0 = P_2(G/H; Z)$.

2. Subgroups of SU(n) with $P_1(\mathrm{SU}(n)/H; Z) = 0 = P_2(\mathrm{SU}(n)/H; Z)$. Let G be a compact Lie group, H a closed connected subgroup, and let T be a maximal torus in H. Then the natural map $\pi: G/T \to G/H$ is a fibre map with fibre H/T, and hence the kernel of the map $\pi^*: H^*(G/H; Z) \to H^*(G/T; Z)$ consists only of torsion elements. Let ξ be the principal bundle $(H \to G \to G/H)$ and ξ' the principal bundle $(T \to G \to G/T)$. Then if ϕ is a real representation of H, $\pi^!(\alpha_\xi(\phi)) = \alpha_{\xi'}(\phi|T)$ and hence $\pi^*(P_j(\alpha_\xi(\phi); Z)) = P_j(\alpha_{\xi'}(\phi|T); Z)$. Let $\eta(G) = (G \to E_G \to B_G)$ be a classifying bundle for G. Then $\eta(T) = (T \to E_G \to B_T)$ is a classifying bundle for T. Letting $i: G/T \to B_T$ be the classifying map of ξ', we have that $\pi^!(\alpha_\xi(\phi)) = i^!(\alpha_{\eta(T)}(\phi|T))$, and hence from [2] that

$$\pi^*(P_j(\alpha_\xi(\phi); Z)) = i^*(P_j(\alpha_{\eta(T)}(\phi|T); Z))$$
$$= (-1)^j i^* \sigma_{2j}(\tau_{\eta(T)}(W_1), \cdots, \tau_{\eta(T)}(W_N), -\tau_{\eta(T)}(W_1), \cdots, -\tau_{\eta(T)}(W_N))$$

where σ_{2j} is the $2j$th elementary symmetric function, $\tau_{\eta(T)}$ is the transgression in the Serre spectral sequence $E(\eta(T))$ associated with $\eta(T)$, and $\{W_1, \cdots, W_N\}$ are the positive weight vectors of ϕ.

Recall now that $\tau(G/H) = \alpha_\xi(\iota_{G/H}) = \alpha_\xi(\mathrm{Ad}_G|H - \mathrm{Ad}_H)$. By Proposition 1.1, $\alpha_\xi(\mathrm{Ad}_G|H)$ is a trivial bundle, and hence $\tau(G/H) \oplus \alpha_\xi(\mathrm{Ad}_H) = \theta$ where θ is a trivial vector bundle.

Corollary 2.1 [2]. *Let T be a torus in G. Then G/T is stably parallelizable.*

From $\pi^!(\tau(G/H)) = i^!\alpha_{\eta(T)}(\iota_{G/H}|T)$ and $\pi^!(\alpha_\xi(\mathrm{Ad}_H)) = i^!\alpha_{\eta(T)}(\mathrm{Ad}_H|T)$, we have the following corollary.

Corollary 2.2. $\Sigma_{\alpha+\beta=j}\, \pi^*(P_\alpha(G/H; Z)) \cdot \pi^*(P_\beta(\alpha_\xi(\mathrm{Ad}_H): Z)) = 0.$

As E_G is contractible, $E_G \times_T G \to G/T$ is a homotopy equivalence, and hence we may take $\gamma = (G \to G/T \xrightarrow{i} B_T)$ to be a principal G bundle. Thus we see that $i^*: H^*(B_T; Z) \to H^*(G/T; Z)$ is just the edge homomorphism

$$H^*(B_T; Z) = E_2^{*,0}(\gamma) \twoheadrightarrow E_\infty^{*,0}(\gamma) \subset H^*(G/T; Z),$$

and hence $\ker i^* = \langle \mathrm{im}\, d(\gamma)^+ \rangle$, the ideal in $H^*(B_T; Z)$ generated by the elements of positive degree in the image of the differential of $E(\gamma)$.

Suppose that U is a closed connected normal subgroup of H. Then by using Proposition 1.1 together with a standard argument, it follows that $P_j(G/H; Z) = 0$ implies that $P_j(G/U; Z) = 0$.

So to determine those closed connected subgroups H of G with $P_1(G/H; Z) = 0 = P_2(G/H; Z)$, we need only determine those closed simple subgroups U of G with $P_1(G/U; Z) = 0 = P_2(G/U; Z)$ and then, by using the classification theorem of compact Lie groups, determine whatever restrictions there are on "piecing" the various simple (we shall see later that if $P_1(\mathrm{SU}(n)/H; Z) = 0$, then H is either semisimple or a torus) groups together to form an acceptable H.

We now let $G = \mathrm{SU}(n)$ and let $\lambda : H \longrightarrow \mathrm{SU}(n)$ be the embedding of H as a closed subgroup, i.e. λ is an injective homomorphism of the compact connected Lie group H into $\mathrm{SU}(n)$. By examining the spectral sequence $E(y)$ of y, we see that in low dimensions at least, $\ker i^* = \langle(\mathrm{im}\, \tau_y)^+\rangle$ where τ_y is the transgression of y. Thus we have (at least in low dimensions)

$$\ker i^* = \langle S^+(\tau_{\eta(T)}(\mu_1), \cdots, \tau_{\eta(T)}(\mu_n))\rangle$$

where $\{\mu_1, \cdots, \mu_n\}$ is the set of weight vectors of λ, and $S^+(x_1, \cdots, x_n)$ is the ring of symmetric polynomials with zero constant term.

We now apply the above to the representation $\phi = \mathrm{Ad}_H$. Note that as $\lambda : H \longrightarrow \mathrm{SU}(n)$, $\mu_n = -(\mu_1 + \cdots + \mu_{n-1})$, and hence

$$2\sigma_2(\tau_{\eta(T)}(\mu_1), \cdots, \tau_{\eta(T)}(\mu_n)) = -\sum_{i=1}^{n} (\tau_{\eta(T)}(\mu_i))^2.$$

Recall that, by Corollary 2.2,

$$\begin{aligned}\pi^*(P_1(\mathrm{SU}(n)/H; Z)) &= -\pi^* P_1(\alpha_\xi(\mathrm{Ad}_H); Z)\\ &= i^*(\sigma_2(\tau_{\eta(T)}(W_1), \cdots, \tau_{\eta(T)}(W_N), -\tau_{\eta(T)}(W_1), \cdots, -\tau_{\eta(T)}(W_N)))\\ &= -i^* \sum_{j=1}^{N} (\tau_{\eta(T)}(W_j))^2.\end{aligned}$$

So $\pi^*(P_1(\mathrm{SU}(n)/H; Z)) = 0$ if and only if for some integer K,

$$K \sum_{j=1}^{n} (\tau_{\eta(T)}(\mu_j))^2 = 2 \sum_{j=1}^{N} (\tau_{\eta(T)}(W_j))^2$$

So if U is a closed, simple, normal subgroup of H, it follows that $\pi^*(P_1(\mathrm{SU}(n)/U; Z)) = 0$ if and only if there exists an integer K such that

$$K \sum_{j=1}^{n} |\mu_j|^2 = 2 \sum_{j=1}^{N} |W_j|^2$$

where $|\ |$ is the Cartan-Killing norm.

We are now ready to determine those closed simple subgroups G of $\mathrm{SU}(n)$ for which $P_1(\mathrm{SU}(n)/G; Z) = 0$. First note that if $\lambda : G \longrightarrow \mathrm{SU}(n)$, then $\mathrm{SU}(n)/\lambda[G]$ is diffeomorphic to $\mathrm{SU}(n)/\lambda^*[G]$. Hence in the following theorem if $\lambda : G \longrightarrow \mathrm{SU}(n)$ is such that $P_1(\mathrm{SU}(n)/\lambda[G]; Z) = 0$, then we shall either include (G, λ) or (G, λ^*) in the list, but not both.

Theorem 2.3. *Suppose* $\lambda : G \to \mathrm{SU}(n)$ *is an almost faithful embedding, where* G *is a simply connected, compact, simple Lie group such that* $P_1(\mathrm{SU}(n)/\lambda[G]; Z) = 0$. *Then modulo trivial representations,* (G, λ) *appears on the following list:*

G	λ	
SU(4)	$b\Lambda^2\mu_4$	$b = 1, 2, 4$
	$a\mu_4 \oplus 3\Lambda^2\mu_4 \oplus d\mu_4^*$	$a + d = 2$
SU(5)	$\mu_5 \oplus b\Lambda^2\mu_5 \oplus c\Lambda^2\mu_5^*$	$b + c = 3$
SU(6)	$b\Lambda^2\mu_6 \oplus c\Lambda^2\mu_6^*$	$b + c = 1, 3$
	$b\Lambda^3\mu_6$	$b = 1, 2$
	$a\mu_6 \oplus b\Lambda^2\mu_6 \oplus \Lambda^3\mu_6 \oplus c\Lambda^2\mu_6^* \oplus d\mu_6^*$	$a + d + 4(b + c) = 6$
SU(7)	$a\mu_7 \oplus b\Lambda^2\mu_7 \oplus c\Lambda^2\mu_7^* \oplus d\mu_7^*$	$a + d = 4,\ b + c = 1$
SU(8)	$\mu_8 \oplus b\Lambda^3\mu_8 \oplus c\Lambda^3\mu_8^*$	$b + c = 1$
SU(k), $k \geq 2$	$\mathrm{Ad}_{\mathrm{SU}(k)}$	
SU(k), $k \geq 2$	$a\mu_k \oplus d\mu_k^*$	$a + d \mid 2k$
SU(k), $k \geq 2$	$a\mu_k \oplus bS^2\mu_k \oplus cS^2\mu_k^* \oplus d\mu_k^*$	$a + d = k - 2,\ b + c = 1$
SU(k), $k \geq 4$	$a\mu_k \oplus b\Lambda^2\mu_k \oplus c\Lambda^2\mu_k^* \oplus d\mu_k^*$	$a + d = 2 + k,\ b + c = 1$
		$a + d = 4,\ b + c = 2$
		$a + d = 2,\ b + c = 1$
SU(k), $k \geq 4$	$\Lambda^2\mu_k \oplus S^2\mu_k^*$	
$\mathrm{S_p}(3)$	$a\Lambda^2\nu_6 \oplus b(\Lambda^3\nu_6 - \nu_6)$	$a + b = 1, 2$
	$4\nu_6 \oplus (\Lambda^3\nu_6 - \nu_6)$	
$\mathrm{S_p}(k)$, $k \geq 3$	$a\nu_{2k}$	$a \mid 2(k + 1)$
$\mathrm{S_p}(k)$, $k \geq 3$	$4\nu_{2k} \oplus \Lambda^2\nu_{2k}$	
$\mathrm{S_p}(k)$, $k \geq 3$	$\mathrm{Ad}_{\mathrm{S_p}(k)}$	
$\mathrm{S_{PIN}}(5)$	$a\rho_5 \oplus b\Delta_5$	$2a + b \mid 6$
$\mathrm{S_{PIN}}(7)$	$a\rho_7 \oplus b\Delta_7$	$a + b \mid 5$
$\mathrm{S_{PIN}}(9)$	$a\rho_9 \oplus b\Delta_9$	$a + 2b \mid 7$
$\mathrm{S_{PIN}}(11)$	$a\rho_{11} \oplus b\Delta_{11}$	$a + 4b \mid 9$

$\mathrm{S_{PIN}}(13)$	$a\rho_{13} \oplus b\Delta_{13}$	$a+8b \mid 11$
$\mathrm{S_{PIN}}(2k+1),\ k \geq 2$	$a\rho_{2k+1}$	$a \mid (2k-1)$
$\mathrm{S_{PIN}}(2k+1),\ k \geq 2$	$\mathrm{Ad}_{\mathrm{S_{PIN}}(2k+1)}$	
$\mathrm{S_{PIN}}(8)$	$a\rho_8 \oplus b\Delta_8^+ \oplus c\Delta_8^-$	$a+b+c \mid 6$
$\mathrm{S_{PIN}}(10)$	$a\rho_{10} \oplus b\Delta_{10}^+ \oplus c\Delta_{10}^-$	$a+2(b+c) \mid 8$
$\mathrm{S_{PIN}}(12)$	$a\rho_{12} \oplus b\Delta_{12}^+ \oplus c\Delta_{12}^-$	$a+4(b+c) \mid 10$
$\mathrm{S_{PIN}}(14)$	$a\rho_{14} \oplus b\Delta_{14}^+ \oplus c\Delta_{14}^-$	$a+8(b+c) \mid 12$
$\mathrm{S_{PIN}}(2k),\ k \geq 4$	$a\rho_{2k}$	$a \mid 2(k-1)$
$\mathrm{S_{PIN}}(2k),\ k \geq 4$	$\mathrm{Ad}_{\mathrm{S_{PIN}}(2k)}$	
G_2	$\phi_2 = \mathrm{Ad}_{G_2}, \phi_1, 2\phi_1, 4\phi_1$	
F_4	$\phi_4 = \mathrm{Ad}_{F_4}, \phi_1, 3\phi_1$	
E_6	$a\phi_1 \oplus b\phi_5$	$a+b=1,2,4$
	$\mathrm{Ad}_{E_6} = \phi_6$	
E_7	$\phi_1, 3\phi_1, \mathrm{Ad}_{E_7} = \phi_6$	
E_8	Ad_{E_8}	

Remark. Note that if $\overline{G}$ is a compact simple Lie group and $\lambda : \overline{G} \to G_1$ is any nontrivial homomorphism of Lie groups, then Ker λ is a finite central subgroup. So any embedding of $\overline{G}$ lifts to an almost faithful embedding of $\overline{G}$'s universal covering group $\tilde{\lambda} : \tilde{G} \to G_1$. If $G_1 = \mathrm{SU}(n)$ and $P_1(\mathrm{SU}(n)/\overline{G}) = 0$, then $(\tilde{G}, \tilde{\lambda})$ will be on the above list.

Proof of Theorem 2.3. Suppose $G = \mathrm{SU}(k)$, $2 \leq k \leq n$. If $\lambda : \mathrm{SU}(k) \to \mathrm{SU}(n)$ and $\Sigma(\lambda) = \{(j, \mu_j) : 1 \leq j \leq n\}$ is the set of weight vectors of λ, let $n(\lambda) = \Sigma_{j=1}^n |\mu_j|^2$. For $1 \leq p,\ q \leq k$, let e_q^p be the $k \times k$ matrix $e_q^p = [\delta_p^i \delta_q^j]$. Let $\lambda_j = i(e_j^j - k^{-1} \Sigma_{t=1}^k e_t^t)$ for $1 \leq j \leq k$. Then $\{\lambda_j : 1 \leq j \leq k-1\}$ forms a basis for $\mathfrak{h}$, the standard Cartan subalgebra of $\mathrm{SU}(k)$. We may take our invariant, nondegenerate inner-product for the real Lie algebra of $\mathrm{SU}(k)$ so that $(\lambda_i, \lambda_i) = (k-1)/k$ and $(\lambda_i, \lambda_j) = -1/k$ if $i \neq j$. Note that

$$n(\mathrm{Ad}_{\mathrm{SU}(k)}) = 2 \sum_{1 \leq i < j \leq k} |\lambda_i - \lambda_j|^2 = 2 \sum_{1 \leq i < j \leq k} 2 = 2k(k-1).$$

So as we have shown, $P_1(\mathrm{SU}(n)/\lambda[\mathrm{SU}(k)]) = 0$ implies $2k(k-1)$ is an integral multiple of $n(\lambda)$.

We let $\pi = \{\lambda_j - \lambda_{j+1} : 1 \leq j \leq k-1\}$ be our system of simple roots. If ϕ :

$SU(k) \to SU(n)$ is irreducible, let Λ_ϕ be the highest weight of ϕ, and let

$$q_j = \frac{2(\Lambda_\phi, \lambda_j - \lambda_{j+1})}{(\lambda_j - \lambda_{j+1}, \lambda_j - \lambda_{j+1})} = (\Lambda_\phi, \lambda_j - \lambda_{j+1}).$$

Then recall q_j is a nonnegative integer.

For each $1 \le i \le k-1$, let ϕ_i be the unique irreducible representation satisfying

$$\frac{2(\Lambda_{\phi_i}, \lambda_j - \lambda_{j+1})}{(\lambda_j - \lambda_{j+1}, \lambda_j - \lambda_{j+1})} = \delta^i_j, \quad \text{i.e. } \phi_i \leftrightarrow \circ\!-\!\circ \cdots \circ\!-\!\overset{1}{\circ}\!-\!\circ \cdots \circ\!-\!\circ$$

Recall that $\phi_1 = \mu_k$ and $\phi_j = \Lambda^j \mu_k$, $1 \le j \le k$. So

$$\sum(\phi_j) = \{\lambda_{i_1} + \cdots + \lambda_{i_j} : 1 \le i_1 < i_2 < \cdots < i_j \le k\}.$$

Hence

$$n(\phi_j) = \sum_{1 \le i_1 < \cdots < i_j \le k} |\lambda_{i_1} + \cdots + \lambda_{i_j}|^2 = \sum_{1 \le i_1 < \cdots < i_j \le k} j\left(\frac{k-j}{k}\right)$$

$$= \frac{j}{k}(k-j)\frac{k(k-1)\cdots(k-j+1)}{1 \cdot 2 \cdots j}$$

$$= \frac{(k-1)(k-2)\cdots(k-j+1)}{2 \cdot 3 \cdots j}\, j(k-j) \quad \text{if } j > 1$$

$$= \frac{(k-1)(k-2)\cdots(k-j+2)}{2 \cdot 3 \cdots (j-1)}\,(k-j+1)(k-j) \quad \text{if } j > 2.$$

Note that $n(\phi_{j+1}) = n(\phi_j) \cdot (k-j-1)/j$. So if $1 \le j \le (k-1)/2$, then $n(\phi_{j+1}) \ge n(\phi_j)$. Also, $1 \le j \le (k-1)/2$ implies $\Sigma(\phi_j) = -\Sigma(\phi_{k-j})$ (i.e. $\phi_j = \phi^*_{k-j}$) which implies $n(\phi_j) = n(\phi_{k-j})$. We have

$$n(\phi_1) = k-1, \qquad n(\phi_2) = (k-1)(k-2),$$

$$n(\phi_3) = \tfrac{1}{2}(k-1)(k-2)(k-3) > 2k(k-1) \quad \text{if } k > 9.$$

So if $4 \le j \le [(k+1)/2]$ and $n(\phi_j) \le 2k(k-1)$, it follows that $k \le 8$, and hence $k = 8$, $j = 4$. By a direct check, this is impossible. So if $1 \le j \le [(k+1)/2]$ and $n(\phi_j) \le 2k(k-1)$, we have $j = 1$, $j = 2$, or $j = 3$ and $6 \le k \le 8$.

Lemma 2.4. *Suppose* $\phi : SU(k) \to SU(n)$ *is an irreducible map with* $n(\phi) \le 2k(k-1)$. *Then up to conjugation,* ϕ *appears on the following list:*

$$\Lambda^3\mu_k, \qquad 6 \leq k \leq 8,$$

$$\mu_k, S^2\mu_k, \Lambda^2\mu_k, \mathrm{Ad}_{\mathrm{SU}(k)}, \qquad k \geq 2.$$

Proof. Let

$$\phi \leftrightarrow \overset{q_1}{\circ}\!-\!\overset{q_2}{\circ}\!-\!\overset{q_3}{\circ}\cdots\overset{q_{k-1}}{\circ}.$$

Suppose that there exist at least three distinct indices a, b, c so that q_a, q_b, q_c all are different from zero. If ϕ_1, ϕ_2 are two different irreducible representations, we say $\phi_1 \geq \phi_2$ if $q_i(\phi_1) \geq q_i(\phi_2)$ for $i = 1, 2, \cdots, k-1$. It follows easily that $\phi_1 \geq \phi_2$ implies $(\Lambda_{\phi_1}, \Lambda_{\phi_1}) \geq (\Lambda_{\phi_2}, \Lambda_{\phi_2})$.

Using this fact, one easily shows $(\Lambda_\phi, \Lambda_\phi) > 1$.

Since q_a, q_b, q_c are all different from zero, it follows easily by applying the Weyl group that $\Sigma(\phi)$ contains at least $(k-1)(k-2)(k-3)$ distinct vectors Λ all with $(\Lambda, \Lambda) = (\Lambda_\phi, \Lambda_\phi)$. So $n(\phi) \geq (k-1)(k-2)(k-3)$. So if $k > 6$, $n(\phi) > 2k(k-1)$.

Note $4 \leq k$ as there exist $q_a, q_b, q_c \neq 0$. If $4 \leq k \leq 6$, one shows by direct calculation that $(\Lambda_\phi, \Lambda_\phi) \geq 5$ and hence

$$n(\phi) \geq 5(k-1)(k-2)(k-3) > 2k(k-1) \quad \text{if } 4 \leq k \leq 6.$$

Suppose next that there exist exactly 2 indices a, b with $q_a, q_b \neq 0$. Suppose $k > 9$. Further suppose that $3 \leq a \leq k-3$. Then $\phi \geq \Lambda^a\mu_k = \phi_a$. So $(\Lambda_\phi, \Lambda_\phi) \geq (\Lambda_{\phi_a}, \Lambda_{\phi_a})$. Also, by applying the Weyl group, it is clear that we introduce at least as many new vectors Λ into $\Sigma(\phi)$ with $(\Lambda, \Lambda) = (\Lambda_\phi, \Lambda_\phi)$, as we do vectors Λ^1 into $\Sigma(\Lambda^a\mu_k)$ with $(\Lambda^1, \Lambda^1) = (\Lambda_{\phi_a}, \Lambda_{\phi_a})$, and so $n(\phi) \geq n(\Lambda^a\mu_k) > 2k(k-1)$ (as $3 < a < k-3$).

It follows that we may assume (up to conjugation)

(1) $a = 1$, $b = 2$,

(2) $1 \leq a \leq 2$, $k-2 \leq b \leq k-1$,

(3) $1 \leq a \leq 2$, $b = 3$, $6 \leq k \leq 8$,

(4) $1 \leq a \leq 3$, $k-3 \leq b \leq k-1$, $7 \leq k \leq 8$.

By direct computation, we now easily arrive at the result.

With this lemma, the results with $G = \mathrm{SU}(k)$ follow easily. One obtains the results for $G = B_k, C_k, D_k, G_2, F_4, E_6, E_7, E_8$ similarly.

Theorem 2.5. *Suppose $\lambda: G \to \mathrm{SU}(n)$ is a nontrivial representation, where G is a simply connected, compact, simple Lie group. Suppose $P_1(\mathrm{SU}(n)/\lambda[G]; Z) = 0 = P_2(\mathrm{SU}(n)/\lambda[G]; Z)$. Then (G, λ) (modulo an arbitrary trivial representation) appears on the following list:*

G	λ	
SU(2)	$4\mu_2$	
SU(3)	$\mu_3 \oplus bS^2\mu_3 \oplus cS^2\mu_3^*$	$b + c = 1$
SU(4)	$\Lambda^2\mu_4$	
	$2\Lambda^2\mu_4$	
	$\Lambda^2\mu_4 \oplus S^2\mu_4^*$	
SU(6)	$\Lambda^2\mu_6$	
SU(k), $k \geq 2$	$\mathrm{Ad}_{\mathrm{SU}(k)}$	
SU(k), $k \geq 2$	$a\mu_k \oplus b\mu_k^*$	$a + b = 1$
		$a + b = 2$
		$a + b = 3,\ 3 \mid k$
		$a + b = 6,\ 3 \mid k$
$\mathrm{S_p}(3)$	$\Lambda^2\nu_6$	
$\mathrm{S_p}(k)$, $k \geq 3$	ν_{2k}	
	$3\nu_{2k}$	$3 \mid k + 1$
	$\mathrm{Ad}_{\mathrm{S_p}(k)}$	
$\mathrm{S_{PIN}}(5)$	$a\rho_5 \oplus b\Delta_5$	$2a + b \mid 6,\quad b \neq 4$
$\mathrm{S_{PIN}}(2k+1)$, $k \geq 2$	ρ_{2k+1}	
	$3\rho_{2k+1}$	$3 \mid 2k - 1$
	$\mathrm{Ad}_{\mathrm{S_{PIN}}(2k+1)}$	
$\mathrm{S_{PIN}}(8)$	$a\rho_8 \oplus b\Delta_8^+ \oplus c\Delta_8^-$	$a + b + c \mid 6,\quad b + c > 0$
$\mathrm{S_{PIN}}(10)$	Δ_{10}^+	
$\mathrm{S_{PIN}}(2k)$, $k \geq 4$	ρ_{2k}	
	$2\rho_{2k}$	
	$3\rho_{2k}$	$3 \mid k - 1$
	$6\rho_{2k}$	$3 \mid k - 1$
	$\mathrm{Ad}_{\mathrm{S_{PIN}}(2k)}$	

G_2	$\phi_2 = \mathrm{Ad}_{G_2}, \phi_1$
F_4	$\phi_4 = \mathrm{Ad}_{F_4}, \phi_1$
E_6	Ad_{E_6}
E_7	Ad_{E_7}
E_8	Ad_{E_8}

Proof. Let $\{(j, \mu_j): 1 \leq j \leq n\} = \Sigma(\lambda)$, and let $\sigma_j(\lambda) = \sigma_j(r_{\eta(T)}(\mu_1), \cdots, r_{\eta(T)}(\mu_n))$ for $2 \leq j \leq 4$. Recall that

$$(\ker i^*)^8 = (Zv_1 \oplus \cdots \oplus Zv_l)^2 \sigma_2(\lambda) \oplus (Zv_1 \oplus \cdots \oplus Zv_l)\sigma_3(\lambda)$$

$$\oplus Z\sigma_4(\lambda) \quad \text{where } l = \text{rank } G.$$

Now

$$P_2(\mathrm{SU}(n)/\mathrm{im}\,\lambda; Z) + P_1(\mathrm{SU}(n)/\mathrm{im}\,\lambda; Z) \cdot P_1(\alpha_\xi(\mathrm{Ad}_U); Z) + P_2(\alpha_\xi(\mathrm{Ad}_U); Z) \equiv \dot{0}$$

(modulo 2 torsion). So as $P_1(\mathrm{SU}(n)/\mathrm{im}\,\lambda; Z) = 0 = P_2(\mathrm{SU}(n)/\mathrm{im}\,\lambda; Z)$ by hypothesis, we see that $\pi^*(P_1(\alpha_\xi(\mathrm{Ad}_U); Z)) = 0 = \pi^*(P_2(\alpha_\xi(\mathrm{Ad}_U); Z))$. Hence

$$\left(\sum_{j=1}^{l} a_j v_j^2 + \sum_{1 \leq i < j \leq l} b_{ij} v_i v_j\right)\sigma_2(\lambda) + \left(\sum_{j=1}^{l} d_j v_j\right)\sigma_3(\lambda) + m \cdot \sigma_4(\lambda) = \sigma_4(\mathrm{Ad}_G)$$

where $a_j, b_{ij}, d_j, m \in Z$ for $1 \leq i \leq l$, $1 \leq j \leq l$. Suppose $G = \mathrm{SU}(k)$. By using the two facts

$$\sigma_4(y_1, -y_1, \cdots, y_p, -y_p) = \sigma_2(y_1^2, \cdots, y_p^2)$$

and

$$\sigma_j(X_1, \cdots, X_p, Y_1, \cdots, Y_q) = \sum_{\alpha+\beta=j} \sigma_\alpha(X_1, \cdots, X_p) \cdot \sigma_\beta(Y_1, \cdots, Y_q),$$

it follows that

$$\begin{aligned}\sigma_4(\mathrm{Ad}_{\mathrm{SU}(k)}) = (2k^2 - k - 6)\Big(&\sigma_1(v_1^4, \cdots, v_{k-1}^4) + 3\sigma_2(v_1^2, \cdots, v_{k-1}^2) \\ &+ 2\sum_{j=1}^{k-1} v_j^2 \sigma_2(v_1, \cdots, \hat{v}_j, \cdots, v_{k-1})\Big) \\ &+ (8k^2 - 6k - 24)\sum_{j=1}^{k-1} v_j \sigma_3(v_1, \cdots, \hat{v}_j, \cdots, v_{k-1}) \\ &+ (12k^2 - 12k - 36)\sigma_4(v_1, \cdots, v_{k-1})\end{aligned}$$

where $v_j = r_{\eta(T)}(\lambda_j)$ as usual.

As an example of the method, consider $\lambda = a\mu_k \oplus b\mu_k^*$, $a + b|2k$. One shows easily that

$$\sigma_2(\mu_k) = -(\sigma_1(v_1^2, \cdots, v_{k-1}^2) + \sigma_2(v_1, \cdots, v_{k-1})),$$

$$\sigma_3(\mu_k) = -\left(\sum_{j=1}^{k-1} v_j^2 \sigma_1(v_1, \cdots, \hat{v}_j, \cdots, v_{k-1}) + 2\sigma_3(v_1, \cdots, v_{k-1})\right),$$

$$\sigma_4(\mu_k) = -\left(\sum_{j=1}^{k-1} v_j^2 \sigma_2(v_1, \cdots, \hat{v}_j, \cdots, v_{k-1}) + 3\sigma_4(v_1, \cdots, v_{k-1})\right),$$

$$\sigma_4(\lambda) = (a + b)\sigma_4(\mu_k) + \tfrac{1}{2}(a + b)(a + b - 1)(\sigma_2(\mu_k))^2.$$

So

$$\begin{aligned}\sigma_4(\mathrm{Ad}_{SU(k)}) = -\Bigg(&\sum_{j=1}^{k-1} a_j v_j^2 + \sum_{1 \le i < j \le k-1} b_{ij} v_i v_j\Bigg) \\ &\cdot (a + b)(\sigma_1(v_1^2, \cdots, v_{k-1}^2) + \sigma_2(v_1, \cdots, v_{k-1})) \\ &-(a - b)\sum_{j=1}^{k-1} d_j v_j \left(\sum_{j=1}^{k-1} v_j^2 \sigma_1(v_1, \cdots, \hat{v}_j, \cdots, v_{k-1}) + 2\sigma_3(v_1, \cdots, v_{k-1})\right) \\ &+ m\Bigg(\frac{1}{2}(a + b)(a + b - 1)\sigma_1(v_1, \cdots, v_{k-1}) \\ &\quad + \frac{3}{2}(a + b)(a + b - 1)\sigma_2(v_1^2, \cdots, v_{k-1}^2) \\ &\quad + (a + b)(a + b - 1)\sum_{j=1}^{k-1} v_j^3 \sigma_1(v_1, \cdots, \hat{v}_j, \cdots, v_{k-1}) \\ &\quad + (2(a + b)(a + b - 1) - (a + b))\sum_{j=1}^{k-1} v_j^2 \sigma_2(v_1, \cdots, \hat{v}_j, \cdots, v_{k-1}) \\ &\quad + (3(a + b)(a + b - 1) - 3(a + b))\sigma_4(v_1, \cdots, v_{k-1})\Bigg).\end{aligned}$$

Clearly, $a_i = a_j$ and $d_i = d_j$ if $1 \le i, j \le k - 1$. Also, $b_{ij} = b_{\alpha\beta}$ if $1 \le i < j \le k - 1$, $1 \le \alpha < \beta \le k - 1$. Note $-(a + b)a_1 + \tfrac{1}{2}(a + b)(a + b - 1)m = 2k^2 - k - 6$.

Suppose $k = 2$. Then this is the only condition. So $-(a + b)a_1 + \tfrac{1}{2}(a + b)(a + b - 1)m = 0$. This is clearly acceptable for $a + b|4$.

Suppose $k > 2$.

$$-(a + b)(a_1 + a_2 + b_{12}) - (a - b)(d_1 + d_2) + 3(a + b)(a + b - 1)m/2 = 6k^2 - 3k - 18.$$

So $-(a + b)b_{12} - (a - b)(d_1 + d_2) + \tfrac{1}{2}(a + b)(a + b - 1)m = 2k^2 - k = 6$. Also, $-(a + b)(a_1 + b_{12}) - (a - b)d_1 + (a + b)(a + b - 1)m = 4k^2 - 2k - 12$. So $-(a - b)d_2 = 0$.

Hence $-(a+b)b_{12} + \frac{1}{2}(a+b)(a+b-1)m = 2k^2 - k - 6$.

Suppose $k = 3$. Then this is the only condition. $-(a+b)a_1 + \frac{1}{2}(a+b)(a+b-1)m = 9$. So $a+b|6$ is acceptable.

Suppose $k \geq 4$. $-(a+b)(a_1 + b_{12} + b_{13} + b_{23}) + (2(a+b)(a+b-1) - (a+b))m = 8k^2 - 6k - 24$ or $(a+b)m = 2k$. So

$$-(a+b)a_1 + (a+b-1)k = 2k^2 - k - 6,$$

$$-(a+b)a_1 = 2k^2 - (a+b)k - 6 = 2(k^2 - 3) - (a+b)k.$$

Hence $(a+b)|2k^2 - 6$. Recall $(a+b)|2k$. Hence $(a+b)|2k^2$. So $(a+b)|6$. So $a\mu_k \oplus b\mu_k^*$, $k \geq 4$ is acceptable when $(a+b=1)$, $(a+b=2)$, $(a+b=3$ and $3|k)$, $(a+b=6$ and $3|k)$. This concludes the proof.

Remark. Suppose $\lambda: G_1 \times G_2 \to SU(n)$ so that $P_1(SU(n)/\text{im}\,\lambda; Z) = 0$. Recall $\lambda = \Sigma_{j=1}^t \phi_j \otimes \psi_j$ where $\phi_1, \cdots, \phi_t, \psi_1, \cdots, \psi_t$ are either trivial or irreducible.

Let T_1 be a maximal torus for G_1, and T_2 for G_2. Let $v_1, \cdots, v_{\text{rk}\,G_1}$ be a base for $H^2(B_{T_1}; Z)$, and $\bar{v}_1, \cdots, \bar{v}_{\text{rk}\,G_2}$ for $H^2(B_{T_2}; Z)$. For some nonzero integer M, we have

$$\sigma_2(\text{Ad}_{G_1}) + \sigma_2(\text{Ad}_{G_2}) = \sigma_2(\text{Ad}_{G_1 \times G_2}) = M \cdot \sigma_2(\lambda) = M \cdot \sum_{j=1}^t \sigma_2(\phi_j \otimes \psi_j)$$

$$= M \cdot \sum_{j=1}^t (\dim \psi_j)\sigma_2(\phi_j) + (\dim \phi_j)\sigma_2(\psi_j)$$

$$= M \cdot \sum_{j=1}^t (\dim \psi_j)\sigma_2(\phi_j) + M \sum_{j=1}^t (\dim \phi_j)\sigma_2(\psi_j).$$

So as $\sigma_2(\text{Ad}_{G_1})$ and $\sigma_2(\phi_j)$ $(1 \leq j \leq t)$ are polynomials in $v_1, \cdots, v_{\text{rk}\,G_1}$, and $\sigma_2(\text{Ad}_{G_2})$ and $\sigma_2(\psi_j)$ $(1 \leq j \leq t)$ are polynomials in $\bar{v}_1, \cdots, \bar{v}_{\text{rk}\,G_2}$, we have

$$\sigma_2(\text{Ad}_{G_1}) = M \sum_{j=1}^t (\dim \psi_j)\sigma_2(\phi_j), \quad \text{and}$$

(2.6)

$$\sigma_2(\text{Ad}_{G_2}) = M \sum_{j=1}^t (\dim \phi_j)\sigma_2(\psi_j).$$

So as $n(\text{Ad}_G) = 0$ if and only if G is a torus, we have the following.

Lemma 2.7. *Let H be a compact connected subgroup of* SU(n) *so that $P_1(SU(n)/H) = 0$. Then either H is a torus or H is semisimple.*

So we may assume that G_1 and G_2 are simple. So if $P_1(SU(n)/\text{im}\,\lambda; Z) = 0 = P_2(SU(n)/\text{im}\,\lambda; Z)$, then as before

(2.8) $\sigma_4(\mathrm{Ad}_{G_1}) + \sigma_2(\mathrm{Ad}_{G_1})\sigma_2(\mathrm{Ad}_{G_2}) + \sigma_4(\mathrm{Ad}_{G_2}) = \sigma_4(\mathrm{Ad}_{G_1 \times G_2}) \in \ker i^*.$

Theorem 2.9. *Suppose that* SU(n) *acts as a differentiable transformation group on a connected manifold* M, *where* $P_1(M; Z) = 0 = P_2(M; Z)$. *Suppose also that* $\dim M \le n^2 - 8n/3 - 2$. *Let* G *be the connected component of the identity of "the" principal isotropy subgroup of the action. Then (possibly modulo a low-dimensional trivial representation),* G *together with its embedding appears on the following list.*

G		λ		
SU(k),	$k \ge 4$	μ_k		
	$k \ge 6$	$a\mu_k \oplus b\mu_k^*$	$a + b = 2$	
	$k \ge 9$	$a\mu_k \oplus b\mu_k^*$	$a + b = 3$, $3 \mid k$	
	$k \ge 18$	$a\mu_k \oplus b\mu_k^*$	$a + b = 6$, $3 \mid k$	
SO($2k + 1$),	$k \ge 2$	ρ_{2k+1}		
	$k \ge 11$	$3\rho_{2k+1}$		$3 \mid 2k - 1$
$\mathrm{S_p}(k)$,	$k \ge 3$	γ_{2k}		
	$k \ge 8$	$3\gamma_{2k}$		$3 \mid k + 1$
SO($2k$),	$k \ge 4$	ρ_{2k}		
	$k \ge 6$	$2\rho_{2k}$		
	$k \ge 10$	$3\rho_{2k}$		$3 \mid k - 1$
	$k \ge 19$	$6\rho_{2k}$		$3 \mid k - 1$

Proof. Note that $n^2 - 8n/3 - 2 \ge \dim M \ge \dim(\mathrm{SU}(n)/\mathrm{im}\,\lambda) = \dim \mathrm{SU}(n) - \dim G = n^2 - 1 - \dim G$ and so $\dim G > 8n/3$; that is, the dimension of G as a Lie group is greater than $8/3$ times the dimension of λ as a representation.

By Lemma 2.7, G is either a torus or G is semisimple. Suppose G is a torus, say $G = T^k$. Then $k = \dim G > 8n/3$. This is impossible as $\mathrm{rk}\,\mathrm{SU}(n) = n - 1$. So G is semisimple.

We first suppose $G = G_1 \times G_2$ where G_1 and G_2 are compact simple Lie groups. Recall $\lambda = \Sigma_{j=1}^t \phi_j \otimes \psi_j$. By 2.6 if for some j both ϕ_j and ψ_j are nontrivial, then there exists a positive integer M such that $n(\mathrm{Ad}_{G_1}) \ge M \cdot \dim \psi_j \cdot n(\phi_j)$ and $n(\mathrm{Ad}_{G_2}) \ge M \cdot \dim \phi_j \cdot n(\psi_j)$. This turns out to be a severe restriction on ϕ_j and ψ_j. Using this fact together with 2.6 and 2.8 in the usual way, we arrive at the list for $G_1 \times G_2$. We then use this list together with the additional requirement $\dim G > 8n/3$ to prove that G must be simple, and that, in fact, (G,λ) lies on the above list.

3. The orbit types of SU(n) actions on manifolds with vanishing first and second integral Pontrjagin classes.

Theorem 3.1. *Let* SU(n) *act as a differentiable transformation group on a connected manifold M. Suppose that* $P_1(M; Z) = 0 = P_2(M; Z)$, *and that* $\dim M \le n^2 - 8n/3 - 2$. *Let* $y \in M$ *be a regular element and let* $x \in M$. *Then after conjugating* G_y^0 *if necessary, the inclusion* $G_y^0 \subseteq G_x^0 \subseteq SU(n)$ *appears on the following list:*

(1) *If*

$$(G_y^0) = \left(SU(k) \xrightarrow{a\mu_k \oplus b\mu_k^* \oplus (n-(a+b))\theta} SU(n)\right)$$

where $a+b = 1, 2, 3, 6$ *and* $3|k$ *if* $a+b = 3, 6$, *then*

$$(G_y^0 \subseteq G_x^0 \subseteq SU(n)) = \left(SU(k) \xrightarrow{\mu_k \oplus (l-k)\theta} SU(l) \xrightarrow{a\mu_l \oplus b\mu_l^* \oplus (n-(a+b)l)\theta} SU(n)\right).$$

(2) *If*

$$(G_y^0) = \left(SO(2k+1) \xrightarrow{a\rho_{2k+1} \oplus (n-(2k+1)a)\theta} SU(n)\right),$$

where $a = 1, 3$ *and* $3|(2k-1)$ *if* $a = 3$, *then*

$$(G_y^0 \subseteq G_x^0 \subseteq SU(n)) = \left(SO(2k+1) \xrightarrow{\rho_{2k+1} \oplus (l-(2k+1))\theta} SO(2l+1) \xrightarrow{a\rho_{2k+1} \oplus (n-(a(2l+1)))\theta} SU(n)\right).$$

(3) *If*

$$(G_y^0) = \left(S_p(k) \xrightarrow{a\gamma_{2k} \oplus (n-2ak)\theta} SU(n)\right)$$

where $a = 1, 3$ *and* $3|(k-1)$ *if* $a = 3$, *then*

$$(G_y^0 \subseteq G_x^0 \subseteq SU(n)) = \left(S_p(k) \xrightarrow{\gamma_{2k} \oplus 2(l-k)\theta} S_p(l) \xrightarrow{a\gamma_{2l} \oplus (n-2al)\theta} SU(n)\right).$$

(4) *If*

$$(G_y^0) = \left(SO(2k) \xrightarrow{a\rho_{2k} \oplus (n-2ak)\theta} SU(n)\right)$$

where $a = 1, 2, 3, 6$ *and* $3|(k-1)$ *if* $a = 3, 6$, *then*

$$(G_y^0 \subseteq G_x^0 \subseteq SU(n)) = \left(SO(2k) \xrightarrow{\rho_{2k} \oplus (n-2k)\theta} SO(2l) \xrightarrow{a\rho_{2l} \oplus (n-2al)\theta} SU(n)\right).$$

Proof. As an example of the method, we shall give the proof for

$$(G_y^0) = \left(\mathrm{SU}(k) \xrightarrow{a\mu_k \oplus b\mu_k^* \oplus (n-2k)\theta} \mathrm{SU}(n)\right)$$

where $a + b = 2$, $a \geq 1$, and $k \geq 6$.

As y is regular, after conjugating G_x by a suitable element of SU(n), we may suppose $G_y^0 \subseteq G_x^0$. Hence we first must study the lattice of closed connected subgroups $G_y^0 \subseteq H \subseteq \mathrm{SU}(n)$.

If V is a subspace of $\mathbf{C}^n$, let SU(V) be the subgroup of SU(n) which leaves $V^\perp$ fixed pointwise. Then in [5], the following theorem is proven.

Theorem. *Let V be a subspace of $\mathbf{C}^n$ with $\dim V \geq 3$. Let H be a subgroup of* SU(n) *which contains* SU(V). *Then there exists a subspace V_1 of $\mathbf{C}^n$ with $V \subseteq V_1$ so that* SU(V_1) *is a normal subgroup of H.*

We shall determine a similar result for G_y^0. Suppose $\phi: G_y^0 \to H$, $\psi: H \to \mathrm{SU}(n)$, and $\lambda = \phi \circ \psi$. We may take $H = T^c \times U_1 \times \cdots \times U_d$ where T^c is a torus of rank c, U_j is a compact, simply connected, simple Lie group, and where ϕ and ψ have finite kernels. Then $\phi = \phi_T \oplus \phi_1 \oplus \cdots \oplus \phi_d$. Clearly ϕ_T is trivial as SU(k) is simple and T^c is abelian. So we may take $V = U_1 \times \cdots \times U_d$, and $\phi = \phi_1 \oplus \cdots \oplus \phi_d$. Now $\psi = \Sigma_{j=1}^t \psi_{j_1} \otimes \cdots \otimes \psi_{j_d}$ where each ψ_{j_i} is either trivial or irreducible.

Let $(\psi \circ \phi)_j = (\psi_{j_1} \circ \phi_1) \otimes \cdots \otimes (\psi_{j_d} \circ \phi_d)$. Note that $(\psi \circ \phi)_j$ is a nontrivial representation for some j.

For some $1 \leq i \leq m$, $\psi_{j_i} \circ \phi_i : \mathrm{SU}(k) \to \mathrm{SU}(n)$ is nontrivial. Note that the nontrivial components of $\psi_{j_i} \circ \phi_i$ have dimension $\geq k$. Also if there exists $1 \leq \alpha \leq m$ with $i \neq \alpha$ and $\psi_{j_\alpha} \circ \phi_\alpha$ nontrivial, then the nontrivial components of $(\psi \circ \phi)_j$ have dimension $\geq k^2$ which is greater than $2k$.

So as $\psi \circ \phi = a\mu_k \oplus b\mu_k^* \oplus (n - 2k)\theta$ and $a + b = 2$, it is clear that we may assume

(1) $(\psi \circ \phi)_j$ is trivial if $j > 2$.

(2) If $j = 1, 2$, then there exists at most one integer i_j with $1 \leq i_j \leq m$ so that $\psi_{j i_j} \circ \phi_{i_j}$ is nontrivial.

So G_y^0 is contained in a normal subgroup V of H, where $V = U$, or $V = U_1 \times U_2$.

Suppose $V = \mathrm{SU}(l)$. Then we may write $\phi: \mathrm{SU}(k) \to \mathrm{SU}(l)$, $\psi: \mathrm{SU}(l) \to \mathrm{SU}(n)$, and $\psi \circ \phi = \lambda = a\mu_k \oplus b\mu_k^* \oplus (n - 2k)\theta$, where $a + b = 2$. So if we write $\psi = \psi_1 \oplus \cdots \oplus \psi_t \oplus L\theta: \mathrm{SU}(l) \to \mathrm{SU}(n)$, where ψ_j is irreducible for $1 \leq j \leq t$, then $(\psi_1 \circ \phi) \oplus \cdots \oplus (\psi_t \circ \phi) \oplus L\theta = \psi \circ \phi = a\mu_k \oplus b\mu_k^* \oplus (n - 2k)\theta$.

Hence $\psi = \psi_1 \oplus L\theta$ or $\psi = \psi_1 \oplus \psi_2 \oplus L\theta$.

Now $l^2 - 1 \geq k^2 - 1 > 8n/3$. Hence ψ_1 and ψ_2 have drastic limitations on their dimensions. Using the fact that $\phi > \psi$ implies $\dim \phi > \dim \psi$, it follows that the only possible irreducible components of ψ are μ_l and μ_l^*. Arguing similarly for ϕ, we see that the only possible irreducible components of ϕ are μ_k and μ_k^*.

Hence, it is clear that one of the following cases holds:

(1) $\phi = a\mu_k \oplus b\mu_k^* \oplus (l-2k)\theta$, $\psi = \mu_l \oplus (n-l)\theta$.

(2) $\phi = b\mu_k \oplus a\mu_k^* \oplus (l-2k)$, $\psi = \mu_l^* \oplus (n-l)\theta$.

(3) $\phi = \mu_k \oplus (l-k)\theta$, $\psi = a\mu_l \oplus b\mu_l^* \oplus (n-2l)\theta$.

(4) $\phi = \mu_k^* \oplus (l-k)\theta$, $\psi = b\mu_l \oplus a\mu_l^* \oplus (n-2l)\theta$.

Arguing similarly for the case when V is some other simple compact Lie group (and using the fact that $\dim V > k^2 - 1 > 8n/3$), we get the following additional possibilities, where $\phi : \mathrm{SU}(k) \to V$, $\psi : V \to \mathrm{SU}(n)$, and $\psi \circ \phi = \lambda$.

V	ϕ	ψ
$\mathrm{S_p}(l)$ $(a = 1 = b)$	$\mu_k \oplus \mu_k^* \oplus 2(l-k)\theta$	$\gamma_{2l} \oplus (n-2l)\theta$
$\mathrm{SO}(l)$ $(a = 1 = b)$	$\mu_k \oplus \mu_k^* \oplus (l-2k)\theta$	$\rho_l \oplus (n-l)\theta$

Now suppose $V = U_1 \times U_2$. Then using arguments similar to the above, one shows that $V = \mathrm{SU}(l_1) \times \mathrm{SU}(l_2)$, and that we can write $\phi = \phi_1 \oplus \phi_2$ and $\psi = (\psi_1 \otimes \theta) \oplus (\theta \otimes \psi_2)$ where modulo trivial copies,

$$\phi_1 = \mu_k, \mu_k^*, \quad \phi_2 = \mu_k, \mu_k^*,$$

$$\psi_1 = \mu_{l_1}, \mu_{l_1}^*, \quad \psi_2 = \mu_{l_2}, \mu_{l_2}^*.$$

We now specialize to the case $H = G_x^0$.

We shall show that $H = G_x^0$ implies $V = \mathrm{SU}(l)$, and that in fact

$$(G_y^0 \subset V \subset \mathrm{SU}(n)) = \left(\mathrm{SU}(k) \xrightarrow{\mu_k \oplus (l-k)\theta} \mathrm{SU}(l) \xrightarrow{a\mu_l \oplus b\mu_l^* \oplus (n-2k)\theta} \mathrm{SU}(n)\right).$$

Let $\xi_1 = (G_x \to G \to G/G_x)$. Then for $j = 1, 2$,

$$0 = P_j(\tau(G/G_x) \oplus \nu(G/G_x)) = P_j(\alpha_{\xi_1}(\mathrm{Ad}_G | G_x - \mathrm{Ad}_{G_x}) \oplus \alpha_{\xi_1}(\phi_x)).$$

Now V is a connected normal subgroup of G_x^0 so that

$$G_y^0 \xrightarrow{\phi} V \subseteq G_x \subseteq \mathrm{SU}(n) = G$$

i.e. $\psi \circ \phi = \lambda$.

Let $\pi : G/V \to G/G_x$ be the natural map.

Let $\xi_2 = (V \xrightarrow{\psi} G \to G/V)$. Then if Γ is a real representation of G_x,

$\pi^!(\alpha_{\xi_1}(\Gamma)) = \alpha_{\xi_2}(\Gamma|V)$. So

$$\begin{aligned}\pi^!(\alpha_{\xi_1}(\mathrm{Ad}_G | G_x - \mathrm{Ad}_{G_x}) \oplus \alpha_{\xi_1}(\phi_x)) &= \alpha_{\xi_2}(\mathrm{Ad}_G | V - \mathrm{Ad}_{G_x} | V) \oplus \alpha_{\xi_2}(\phi_x | V)\\ &= \alpha_{\xi_2}(\mathrm{Ad}_G | V - \mathrm{Ad}_V - \theta) \oplus \alpha_{\xi_2}(\phi_x | V)\\ &= \tau(G/V) \oplus \alpha_{\xi_2}(\phi_x | V) \in \widetilde{KO}(G/V),\end{aligned}$$

where θ is some trivial representation of V. So modulo torsion,

$$P_1(G/V) + P_1(\alpha_{\xi_2}(\phi_x | V)) = 0,$$

and

$$P_2(G/V) + P_1(G/V) \cdot P_1(\alpha_{\xi_2}(\phi_x | V)) + P_2(\alpha_{\xi_2}(\phi_x | V)) = 0.$$

Recall that as G_y is the principal isotropy subgroup of ϕ_x, $\phi_x|G_y = \mathrm{Ad}_{G_x}|G_y - \mathrm{Ad}_{G_y} \oplus$ trivial copies. So

$$\begin{aligned}\phi_x|G_y^0 &= \mathrm{Ad}_{G_x}|G_y^0 - \mathrm{Ad}_{G_y}|G_y^0 \oplus \text{trivial copies}\\ &= ((\mathrm{Ad}_{G_x}|G_x^0)|V)|G_y^0 - \mathrm{Ad}_{G_y^0} \oplus \text{trivial copies}\\ &= \mathrm{Ad}_V|G_y^0 - \mathrm{Ad}_{G_y^0} \oplus \text{trivial copies}\\ &= \mathrm{Ad}_V \circ \phi - \mathrm{Ad}_{G_y^0} \oplus \text{trivial copies}.\end{aligned}$$

Note also that $\dim(\iota_{\mathrm{SU}(n)/G_x} \oplus \phi_x) = \dim M \leq n^2 - 8n/3 - 2$, and hence $\dim \phi_x \leq n^2 - 8n/3 - 2$.

We shall now examine the several possibilities for V, ϕ, and ψ.

Suppose $V = \mathrm{SU}(l)$, and $\phi = a\mu_k \oplus b\mu_k^* \oplus (l - 2k)\theta$, $\psi = \mu_l \oplus (n - l)\theta$. Then

$$\begin{aligned}\phi_x|\mathrm{SU}(k) = \phi_x \circ \phi &= (\mu_l \otimes \mu_l^* - \theta) \circ \phi - \mu_k \otimes \mu_k^* \oplus \text{trivial copies}\\ &= ab(\mu_k \otimes \mu_k \oplus \mu_k^* \otimes \mu_k^*) \oplus (a^2 + b^2 - 1)\mu_k \otimes \mu_k^*\\ &\quad \oplus (a + b)(l - (a + b)k)(\mu_k \oplus \mu_k^*) \oplus \text{trivial copies}.\end{aligned}$$

Recall $k^2 - 1 > 8n/3$, and $l \geq 2k$. So $\dim \phi_x < n^2 < 9k^4/64 < 9l^4/1024$.

This is a severe limitation on ϕ_x. By listing all the irreducible representations Γ of $\mathrm{SU}(l)$ with $\dim \Gamma < 9l^4/1024$ and then using our determination of $\phi_x \circ \phi$, we see easily that $\phi_x|\mathrm{SU}(l)$ can contain copies only of μ_l, μ_l^*, $\mu_l \otimes \mu_l$, $\mu_l \otimes \mu_l^*$, $\mu_l^* \otimes \mu_l^*$. Hence

$$\phi_x|\mathrm{SU}(l) = c(\mu_l \otimes \mu_l \oplus \mu_l^* \otimes \mu_l^*) \oplus d\mu_l \otimes \mu_l^* \oplus e(\mu_l \oplus \mu_l^*) \oplus F\theta,$$

and so $\phi_x|\mathrm{SU}(k) = \phi_x \circ \phi = (4abc + (a^2 + b^2)\,d)\mu_k \otimes \mu_k^* \oplus$ a sum of representations not containing $\mu_k \otimes \mu_k^*$. So $4abc + (a^2 + b^2)d = a^2 + b^2 - 1$. This is impossible.

In the other cases for V, ϕ, ψ, we argue similarly to the above. In some cases there will exist a representation $\phi_x|V$ so that $\phi_x \circ \phi = \lambda$. In these cases, by using $P_1(G/V) + P_1(\alpha_{\xi_2}(\phi_x|V)) = 0$, we arrive at the conclusion that $V = \mathrm{SU}(l)$, $\phi = \mu_k \oplus (l-k)\theta$, and $\psi = a\mu_l \oplus b\mu_l^*$, i.e.

$$(G_y^0 \subseteq V \subseteq \mathrm{SU}(n)) = \left(\mathrm{SU}(k) \xrightarrow{\mu_k \oplus (l-k)\theta} \mathrm{SU}(l) \xrightarrow{a\mu_l \oplus b\mu_l^* \oplus (n-2k)\theta} \mathrm{SU}(n)\right)$$

as was to be shown.

Finally we shall show $G_x^0 = V$.

So suppose $G_x^0 \neq V$. We shall derive a contradiction. We give here a modification of an argument found in [4].

We may suppose x was chosen so that $\dim G_x$ is minimal among these G_t with G_t^0 not simple. We see that after conjugating if necessary, we have $y \in S_x$, the slice at x. Let ϕ_x be the slice representation of G_x at x. Then $\phi_x|G_x^0 = \phi_1 \oplus \beta\theta$, where ϕ_1 is a nontrivial representation without trivial copies.

We may write $y = \bar{y} + \eta$ where $\bar{y} \in W_{\phi_1}$, the representation space of ϕ_1, and where $\eta \in (W_{\phi_1})^\perp$ i.e. $\eta \in W_{\beta\theta}$.

Clearly $G_y = G_{\bar{y}}$ and so $\bar{y}$ is a regular element.

Let z be an element of the unit ball of $W_{\phi_1} \subseteq S_x \subseteq M$. To show: $(G_x^0(\phi_1)_z)^0 \subseteq V$. By the minimality condition

$$(G_z^0) = \left(\mathrm{SU}(l') \xrightarrow{a\mu_{l'} \oplus b\mu_{l'}^* \oplus (n-2l')\theta} \mathrm{SU}(n)\right).$$

So as $G_z^0 \subseteq G_x^0 \subseteq N_{\mathrm{SU}(n)}(V)$, clearly $G_z^0 \subseteq V$. Hence $(G_x^0(\phi_1)_z)^0 = G_z^0 \subseteq V$. So we may apply the following proposition found in [4].

Proposition. *Let H be a closed, connected, proper normal subgroup of a compact, connected, Lie group K.*

Let $\psi : K \to O(m)$ be a representation, such that for each $x \in W_\psi - \{0\}$, $K_x^0 \subseteq H$. Then

(1) $H \subseteq \ker \psi$.

(2) $\operatorname{rank} K/H = 1$.

Hence $V \subseteq \ker \phi_1 = \ker \phi_x | G_x^0$ and rank $G_x^0/V = 1$. So $\phi_x | G_y^0$ is trivial. But note that $\phi_x \circ \phi = (l-k)(\mu_k \oplus \mu_k^*) \oplus$ trivial copies, as

$$\begin{aligned}\phi_x | G_y^0 &= (\phi_x | G_y) | G_y^0 = (\mathrm{Ad}_{G_x} | G_y - \mathrm{Ad}_{G_y}) | G_y^0 \\ &= (\mathrm{Ad}_{G_x} | V) | G_y^0 - \mathrm{Ad}_{G_y} | G_y^0 \\ &= \mathrm{Ad}_V | G_y^0 - \mathrm{Ad}_{G_y^0} \oplus \theta.\end{aligned}$$

Hence $l = k$, and so $V = G_y^0$. So $G_x^0 \sim G_y^0 \times L$, where either $L = \mathrm{SU}(2)$ or $L = T^1$.

Note that $\phi_x | G_x^0 = \theta \otimes \gamma$. Let $\{a_1\}$ be the standard base for "the" Cartan subalgebra of L. Suppose

$$(G_x^0 \to \mathrm{SU}(n)) = \left(\mathrm{SU}(k) \times L \xrightarrow{(\{a\mu_k \oplus b\mu_k^* \oplus t\theta\} \oplus \Gamma) \otimes (\theta \otimes \delta)} \mathrm{SU}(n)\right).$$

Then $\dim \Gamma = 1$. So $\Gamma = \theta$ or $L = T^1$. Suppose $\Gamma = \theta$. Let $\xi_3 = (G_x^0 \to \mathrm{SU}(n) \to \mathrm{SU}(n)/G_x^0)$. Then

$$P_1(\alpha_{\xi_3}(\iota_{\mathrm{SU}(n)/G_x^0}) \oplus \alpha_{\xi_3}(\phi_x | G_x^0)) = 0.$$

So

$$\begin{aligned}M(2\sigma_2(\mu_k) + \sigma_2(\delta)) &= \sigma_2(\iota_{\mathrm{SU}(n)/G_x^0}) + \sigma_2(\phi_x | G_x^0) \\ &= \sigma_2((((a\mu_k \oplus b\mu_k^* \oplus t\theta) \otimes \theta) \oplus (\theta \otimes \delta)) \\ &\qquad \otimes (((a\mu_k^* \oplus b\mu_k \oplus t\theta) \otimes \theta) \oplus (\theta \otimes \delta^*)) - \mu_k \otimes \mu_k^* - \mathrm{Ad}_L) + \sigma_2(\gamma) \\ &= (6k + 4t + 4 \dim \delta)\sigma_2(\mu_k) + (4k + 2t)\sigma_2(\delta) + 2 \dim \delta \cdot \sigma_2(\delta) - \sigma_2(\mathrm{Ad}_L) + \sigma_2(\gamma).\end{aligned}$$

Hence $2M = 6k + 4t + 4 \dim \delta$. So

$$(3k + 2t + 2 \dim \delta)\sigma_2(\delta) = (4k + 2t)\sigma_2(\delta) + 2 \dim \delta \cdot \sigma_2(\delta) - \sigma_2(\mathrm{Ad}_L) + \sigma_2(\gamma)$$

or

$$-k\sigma_2(\delta) = \sigma_2(\gamma) - \sigma_2(\mathrm{Ad}_L).$$

So as $\sigma_2(\delta)$, $\sigma_2(\gamma)$, $\sigma_2(\mathrm{Ad}_L)$ are all polynomials in $r(a_1)$ with negative coefficients, we see $L = \mathrm{SU}(2)$. So $\sigma_2(\mathrm{Ad}_{\mathrm{SU}(2)}) = \sigma_2(\gamma) + k\sigma_2(\delta)$. An easy check of $RO(\mathrm{SU}(2))$ shows this is impossible.

So $\Gamma \neq \theta$. Hence $L = T^1$. By considering $L \hookrightarrow \mathrm{SU}(n)$, we see $(2k + t)\Gamma \oplus \delta$:

$L \longrightarrow \mathrm{SU}(n)$. We easily have

$$M(2\sigma_2(\mu_k) + \sigma_2((2k+t)\Gamma \oplus \delta)) = M\sigma_2(((a\mu_k \oplus b\mu_k^* \oplus t\theta) \otimes \Gamma) \oplus (\theta \otimes \delta))$$

$$= \sigma_2((((a\mu_k \oplus b\mu_k^* \oplus t\theta) \otimes \Gamma) \oplus (\theta \otimes \delta)) \otimes (((a\mu_k^* \oplus b\mu_k \oplus t\theta) \otimes \Gamma^*) \oplus (\theta \otimes \delta^*)) - \mu_k \otimes \mu_k^*) + \sigma_2(y)$$

$$= \sigma_2((a^2 + b^2 - 1)\mu_k \otimes \mu_k^* \oplus ab(\mu_k \otimes \mu_k \oplus \mu_k^* \otimes \mu_k^*) \oplus (a+b)t(\mu_k \oplus \mu_k^*)$$

$$\oplus (a\mu_k \oplus b\mu_k^* \oplus t\theta) \otimes (\Gamma \otimes \delta^*) \oplus (a\mu_k^* \oplus b\mu_k \oplus t\theta) \oplus (\Gamma^* \otimes \delta) \oplus \theta \otimes (\delta \otimes \delta^*)) + \sigma_2(y)$$

$$= (2k((a+b)^2 - 1) + 2(a+b)t + 2(a+b)\dim \delta)\sigma_2(\mu_k)$$

$$+ \sigma_2((2k+t)(\Gamma \otimes \delta^* \oplus \Gamma^* \otimes \delta) \oplus \theta \otimes (\delta \otimes \delta^*)) + \sigma_2(y)$$

$$= (6k + 4t + 4\dim \delta)\sigma_2(\mu_k) + \sigma_2(((2k+t)\Gamma \oplus \delta) \otimes ((2k+t)\Gamma^* \oplus \delta^*))$$

$$= (6k + 4t + 4\dim \delta)\sigma_2(\mu_k) + (4k + 2t + 2\dim \delta)\sigma_2((2k+t)\Gamma \oplus \delta) + \sigma_2(y).$$

So $2M = 6k + 4t + 4\dim \delta$, or $M = 3k + 2t + 2\dim \delta$. Hence

$$(3k + 2t + 2\dim \delta)\sigma_2((2k+t)\Gamma \oplus \delta) = (4k + 2t + 2\dim \delta)\sigma_2((2k+t)\Gamma \oplus \delta) + \sigma_2(y).$$

So $0 = k\sigma_2((2k+t)\Gamma \oplus \delta) + \sigma_2(y)$. But as $\phi_x | G_x^0 = \theta \otimes y$, we see that $y \otimes \mathbf{C}: L \longrightarrow \mathrm{SU}(n)$, and hence $k\sigma_2((2k+t)\Gamma \oplus \delta)$ and $\sigma_2(y)$ are polynomials in $\tau(a_1)$ with negative coefficients. This is impossible.

The other cases follow similarly.

REFERENCES

1. A. Borel et al., *Seminar on transformation groups*, Ann. of Math. Studies, no. 46, Princeton Univ. Press, Princeton, N. J., 1961.

2. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*. I, Amer. J. Math. 80 (1958), 458–538; II, 81 (1959), 351–382; III, 82 (1960), 491–504. MR 21 #1586; 22 #988; 22 #11413.

3. E. A. Grove, SU(*n*) *actions on manifolds with vanishing first and second integral pontrjagin classes*, Proc. Second Conference on Transformation Groups, Amherst, 1971, Springer, 1972, pp. 324–333.

4. W. C. Hsiang and W. Y. Hsiang, *Differentiable actions of compact connected classical groups*. I, Amer. J. Math. 89 (1967), 705–786; II, Ann. of Math. (2) 92 (1970), 189–223. MR 36 #304; 42 #420.

5. W. Y. Hsiang, *On the principal orbit type and P. A. Smith theory of* SU(*p*) *actions*, Topology 6 (1967), 125–135. MR 34 #5084.

6. R. S. Palais, *The classification of G-spaces*, Mem. Amer. Math. Soc. No. 36 (1960). MR 31 #1664.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF RHODE ISLAND, KINGSTON, RHODE ISLAND 02881